Some Theorems On The Sheaf Of Higher Homotopy Groups

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Summary: In this paper, constructing the sheaf \( H_n \) of higher homotopy groups on a connected and locally path connected topological space, its some characterizations are examined. Let the pairs \((X_1, H_{n_1})\) and \((X_2, H_{n_2})\) be given. If the mapping \( f^* : H_{n_1} \to H_{n_2} \) is a sheaf isomorphism, we show that there exists an isomorphism between the pairs \((X_1, H_{n_1})\) and \((X_2, H_{n_2})\).

Keywords: Higher Homotopy Group, Sheaf of Abelian Groups, Regular Covering Space, Sheaf Isomorphism, Covariant Functor.

Introduction

Let \( X \) be a connected and locally path connected space. Then \( X \) is a path connected and has only one path component, that is \( X \). For an arbitrary fixed point \( c \in X \), we will consider \( X \) as a pointed topological space \((X, c)\) unless otherwise stated. Let \( x \) be any point of \( X \) and \( \pi_n(X, x) \) be higher homotopy group of \( X \) with respect to \( x \) and \( H_n = \bigvee_{x \in X} \pi_n(X, x) \). Clearly, \( H_n \) is a set over \( X \) and the mapping \( \Psi : H_n \to X \) defined by \( \Psi(\sigma_x) = x \) for any \( \sigma_x \in (H_n)_x \subset H_n \) is an onto projection.

We introduce on \( H_n \) a natural topology as follows: Let \( x_0 \) an arbitrary fixed point of \( X \),

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$W = W(x_0)$ be a path connected open neighborhood of $x_0$ and $\sigma_{x_0} = [\alpha]_{x_0}$ be a homotopy class of $(H_n)_{x_0}$. Since $X$ is path connected, there exists a path $\gamma$ with initial point $x_0$ and with terminal point $x$, for every $x \in W$. Therefore, the path $\gamma$ determines an isomorphism

$$\gamma^*: (H_n)_{x_0} \rightarrow (H_n)_x$$

defined by $\gamma^*([\alpha]_{x_0}) = [\beta]_x$ for any

$[\alpha]_{x_0} \in (H_n)_{x_0} \subset H_n$. Let us now define a mapping $s: W \rightarrow H_n$ such that $s(x) = \gamma^*([\alpha]_{x_0}) = [\beta]_x$ for every $x \in W$. If $c \in W$, then we define $s(c) = \gamma^*([\alpha]_c) = [\alpha]_c$, by taking $[\gamma] = [1] \in (H_n)_c$. It is seen that, the mapping $s$ depends on both the homotopy classes $[\alpha]_{x_0}$ and $[\gamma]$. Suppose that the homotopy class $[\gamma]$ is chosen as arbitrary fixed, for each $x \in W$. So, the mapping $s$ depends on only the homotopy class $[\alpha]_{x_0}$. $s$ is well defined and $\Psi = 1_W$. Let us denote the totality of the mapping $s$ defined over $W$ by $\Gamma(W, H_n)$.

Let $B$ be a basis of path connected open neighborhoods for each $x \in X$. Then,

$$T_n = \{s(W): W \in B, \ s \in \Gamma(W, H_n)\}$$

is a topology base on $H_n [4]$. In this topology, the mapping $\Psi$ and $s$ are continuous and $\Psi$ is a local homomorphism. Thus, $(H_n, \Psi)$ is a sheaf over $X$. $(H_n, \Psi)$ (or only $H_n$) is called “The Sheaf $H_n$ of Higher Homotopy Groups over $X$” [6,7]. For any open set $W \subset X$, an element $s$ of $\Gamma(W, H_n)$ is called a section of the sheaf $H_n$ over $W$. The group $(H_n)_x = \pi_n(X, x)$ is called the stalk of the sheaf $H_n$ for any $x \in X$. The set $\Gamma(W, H_n)$ is an abelian group with pointwise addition operation. Thus, the operation $+: H_n \oplus H_n \rightarrow H_n$ is continuous for every stalk of $H_n$ [5]. Moreover, the group $(H_n)_x = \pi_n(X, x)$ is abelian for $n > 1$. Hence, $H_n$ is a sheaf of abelian groups over $X$.

The sheaf $H_n$ satisfies the following properties:

1. Any two stalks of $H_n$ are isomorphic with each other.

2. Let $W_1, W_2 \subset X$ be any open sets, $s_1 \in \Gamma(W_1, H_n)$ and $s_2 \in \Gamma(W_2, H_n)$. If $s_1(x_0) = s_2(x_0)$ for any point $x_0 \in W_1 \cap W_2$, then $s_1 = s_2$ over the whole $W_1 \cap W_2$ [10].

3. Let $W \subset X$ be an open set. Every section over $W$ can be extended to a global section over $X$.

4. Let $x \in X$ be any point and $W=W(x)$ be an open set. Then $\psi^{-1}(W) = \bigcup_{i \in I} s_i(W)$ and $\psi|_{s_i(W)}: s_i(W) \rightarrow W$ is a topological mapping for every $i \in I$. Hence, $W=W(x)$ is evenly covered by $\psi$. Thus, $\Gamma(H_n, \psi)$ is an abelian covering space of $X$ [9].

5. A topological stalk preserving mapping of $H_n$ onto itself is called a sheaf isomorphism or a cover transformation, and the set of all cover transformation of $H_n$ is denoted by $T$. Clearly, $T$ is a group and isomorphic to the group $\Gamma(X, H_n)$. Hence, $\Gamma(H_n)_x \cong \Gamma(X, H_n) \cong T$. Thus, $T$ is transitive and $H_n$ is a regular covering space of $X$ [1].

Characterization

Let $X_1, X_2$ be any connected and locally path connected topological spaces and $H_{n_1}$, $H_{n_2}$
be the corresponding sheaves, respectively. Let us denote these as the pairs \((X_1, H_{n_1})\) and \((X_2, H_{n_2})\).

We begin by giving the following definitions.

**Definition 2.1.** Let \(f^*: H_{n_1} \to H_{n_2}\) be a mapping. If \(f^*\) is continuous, a homomorphism on each stalk of \(H_{n_1}\) and maps every stalk of \(H_{n_1}\) into stalk of \(H_{n_2}\), then it is called a sheaf homomorphism.

Let \(f: X_1 \to X_2\) be a continuous mapping and \(f^*: H_{n_1} \to H_{n_2}\) be a sheaf homomorphism. If \(f^* ((H_{n_1})_{x_1}) \subset (H_{n_2})_{f(x_1)}\) for each \(x_1 \in X_1\), then \(f^*\) is called a stalk preserving homomorphism with respect to \(f\) [2].

**Definition 2.2.** Let \(f^*: H_{n_1} \to H_{n_2}\) be a sheaf homomorphism. If \(f^*\) is homeomorphism then \(f^*\) is called a sheaf isomorphism [8].

**Definition 2.3.** Let the pairs \((X_1, H_{n_1})\) and \((X_2, H_{n_2})\) be given. If

1. The mapping \(f: X_1 \to X_2\) is continuous,
2. The mapping \(f^*: H_{n_1} \to H_{n_2}\) is continuous,
3. The mapping \(f^*: H_{n_1} \to H_{n_2}\) is stalk preserving with respect to \(f\),

then \((f, f^*): (X_1, H_{n_1}) \to (X_2, H_{n_2})\) is called a homomorphism between the pairs \((X_1, H_{n_1})\) and \((X_2, H_{n_2})\).

**Definition 2.4.** Let the pairs \((X_1, H_{n_1})\), \((X_2, H_{n_2})\) and the homomorphism \((f, f^*): (X_1, H_{n_1}) \to (X_2, H_{n_2})\) be given. If the mappings \(f\) and \(f^*\) are homomorphisms, then \((f, f^*)\) is called an isomorphism between the pairs \((X_1, H_{n_1})\) and \((X_2, H_{n_2})\) [3].

**Theorem 2.1.** Let the pairs \((X_1, H_{n_1})\) and \((X_2, H_{n_2})\) be given. If the mapping \(f^*: H_{n_1} \to H_{n_2}\) is given as a sheaf homomorphism, then there exists a unique continuous mapping \(f: X_1 \to X_2\) such that the pair \((f, f^*)\) is a homomorphism between the pairs \((X_1, H_{n_1})\) and \((X_2, H_{n_2})\).

**Proof.** To prove this theorem, we must first find a mapping \(f: X_1 \to X_2\). However, for each \((H_{n_1})_{x_1} \subset H_{n_1}\) there exists a stalk \((H_{n_2})_{x_2} \subset H_{n_2}\) \(\exists f^* \left( (H_{n_1})_{x_1} \right) \subset (H_{n_2})_{x_2}\), since \(f^*\) is stalk preserving. Therefore, to any point \(x_1 \in X_1\) there uniquely corresponds a point \(x_2 \in X_2\). If we denote this correspondence by \(f(x_1) = x_2\), then we obtain a mapping \(f: X_1 \to X_2\).

Let us now show that the mapping \(f\) is continuous. Let \(W \subset f(X_1)\) be an open set. We may be prove that the set \(f^{-1}(W)\) is an open set in \(X_1\). Since \(W\) is an open set in \(X_2\), there exists the arcwise connected open sets \(W_i\) in \(X_2\), \(i \in I\), such that \(W = \bigcup_{i \in I} W_i\). Thus

\[
s(W) = \bigcup_{i \in I} s_i(W_i)
\]

is an open set in \(H_{n_2}\), for a section \(s_i \in \Gamma(W_i, H_{n_2})\). However,
Theorem 2.2. Let the pairs \((X_1, H_{n_1})\), \((X_2, H_{n_2})\) and \((X_3, H_{n_3})\) be given. If the mappings \(f_1: X_1 \to X_2\) and \(f_2: X_2 \to X_3\) are sheaf homomorphisms, then there exists a homomorphism between the pairs \((X_1, H_{n_1})\) and \((X_3, H_{n_3})\) such that \(f = f_2 \circ f_1\).

**Proof.** Since the mappings \(f_1^*, f_2^*\) are continuous, the mapping \(f_2^* \circ f_1^*\) is also continuous. By Theorem 2.1, there is a continuous mapping \(f\) from \(X_1\) into \(X_3\). Clearly \(f_2^* \circ f_1^*\) preserves the stalk with respect to \(f\) and \(f_2^* \circ f_1^*\) is a homomorphism on each stalk. Hence the pair \((f, f_2^* \circ f_1^*)\) is a homomorphism between the pairs \((X_1, H_{n_1})\) and \((X_3, H_{n_3})\). Now, let us show that \(f = f_2 \circ f_1\).

Since, for any stalk \((H_{n_1})_{x_1} \subset H_{n_1}\), there is a stalk \((H_{n_2})_{x_2} \subset H_{n_2}\) \(f_1^*((H_{n_1})_{x_1}) \subset (H_{n_2})_{x_2}\) and for any stalk \((H_{n_2})_{x_2} \subset H_{n_2}\), there is a stalk \((H_{n_3})_{x_3} \subset H_{n_3}\) \(f_2^*((H_{n_2})_{x_2}) \subset (H_{n_3})_{x_3}\),

\[
(f_2^* \circ f_1^*)(((H_{n_1})_{x_1})) = f_2^*(f_1^*((H_{n_1})_{x_1})) \subset f_2^*((H_{n_2})_{x_2}) \subset (H_{n_3})_{x_3},
\]

and \(f(x_1) = x_3\). On the other hand \(f_1(x_1) = x_2\) and \(f_2(x_2) = x_3\) since
Now, we can give the following theorem.

**Theorem 2.3.** Let the pairs \((X_1, H_{n_1})\) and \((X_2, H_{n_2})\) be given. If the mapping \(f^*: H_{n_1} \rightarrow H_{n_2}\) is a sheaf isomorphism, then there exists an isomorphism between the pairs \((X_1, H_{n_1})\) and \((X_2, H_{n_2})\).

**Proof.** It follows from the theorem 2.1 that, there exists a continuous mapping \(f: X_1 \rightarrow X_2\). Let us now show that \(f\) is a bijection. In fact, for any two elements \(x_1, y_1 \in X_1\), if \(f(x_1) = f(y_1) = x_2\), then there is a stalk \((H_{n_2})_{x_2} \subset H_{n_2}\), \(x_2 \in X_2\) \(\exists f^*(((H_{n_1})_{x_1}) = f^*(((H_{n_1})_{y_1}) = (H_{n_2})_{x_2}\). However, this is impossible, since \(f^*\) is one-to-one. Therefore \(x_1 = y_1\). On the other hand, for each stalk \((H_{n_2})_{x_2}\), there exists a stalk \((H_{n_1})_{x_1}\) \(\exists f^*(((H_{n_1})_{x_1}) = (H_{n_2})_{x_2}\) since \(f^*\) is onto. It follows from this reason that, for each \(x_2 \in X_2\), there exists an element \(x_1 \in X_1\) \(\exists f(x_1) = x_2\). Hence \(f\) is a bijection. By Theorem 2.1, there is a continuous mapping \(g: X_2 \rightarrow X_1\), since \(f^{-1}\) is continuous. It is similarly shown that \(g\) is a bijection. On the other hand, it can be shown that \(g = f^{-1}\) Therefore \(f\) is a homeomorphism.

Clearly, \(f^{-1}\) preserves the stalk with respect to \(f\). Thus the pair \((f, f^*)\) is an isomorphism.

Now, let \(C\) be the category of the sheaves of higher homotopy groups and sheaf homomorphisms and \(D\) be the category of the connected and locally path connected topological spaces and continuous mappings. Then, we can define a mapping \(F: C \rightarrow D\) as follows:

For any sheaf \(H_n\) and every morphism \(f^*: H_{n_1} \rightarrow H_{n_2}\), let \(F(H_n) = X\) and \(F(f^*) = f: X_1 \rightarrow X_2\). Then,

1. If \(f^* = 1_{H_{n_1}}\), then \(F(1_{H_{n_1}}) = 1_{x_1}\).
2. If \(f_1^*: H_{n_1} \rightarrow H_{n_2}\) and \(f_2^*: H_{n_2} \rightarrow H_{n_3}\) are two sheaf homomorphisms, then \(F(f_2^* \circ f_1^*) = F(f_2^*) \circ F(f_1^*)\).

Thus, we can state the following theorem.

**Theorem 2.4.** There is a covariant functor from the category of the sheaves of higher homotopy groups and sheaf homomorphisms to the category of the connected and locally path connected topological spaces and continuous mappings.

References

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