Some Theorems On The Sheaf Of Higher Homotopy Groups

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Summary: In this paper, constructing the sheaf H_n of higher homotopy groups on a connected and locally path connected topological space, its some characterizations are examined. Let the pairs (X_1, H_{n_1}) and (X_2, H_{n_2}) be given.If the mapping $f^*: H_{n_1} \to H_{n_2}$ is a sheaf isomorphism, we show that there exists an isomorphism between the pairs (X_1, H_{n_1}) and (X_2, H_{n_2}) .

Keywords: Higher Homotopy Group, Sheaf of Abelian Groups, Regular Covering Space, Sheaf Isomorphism, Covariant Functor.

Yüksek Homotopi Gruplarının Demetleri Üzerine Bazı Teoremler

Özet: Bu çalışmada, irtibatlı lokal eğrisel irtibatlı bir topolojik uzay üzerinde yüksek homotopi gruplarının H_n demeti oluşturularak bazı karakterizasyonları incelenmiştir. (X_1, H_{n_1}) ve (X_2, H_{n_2}) iki çift olsun. Eğer $f^*: H_{n_1} \to H_{n_2}$ bir demet izomorfizmi ise (X_1, H_{n_1}) ve (X_2, H_{n_2}) çiftleri arasında bir izomorfizim olduğu gösterilmiştir.

Anahtar Kelimeler: Yüksek Homotopi Grubu, Abelian Grupların Demeti, Regüler Örtü Uzayı, Demet İzomorfizmi, Kovaryant Funktor.

Introduction

Let X be a connected and locally path connected space. Then X is a path connected and has only are path component, that is X . For an arbitrary fixed point $c \in X$, we will consider X as a pointed topological space (X,c) unless otherwise stated. Let x be any point of X and $\pi_n(X, x)$ be higher homotopy group of X with respect to x and $H_n = \underset{x \in X}{V} \pi_n(X, x)$. Clearly, H_n is a set over X and the mapping $\Psi: H_n \to X$ defined by $\Psi(\sigma_x) = x$ for any $\sigma_x \in (H_n)_x \subset H_n$ is an onto projection.

We introduce on H_n a natural topology as follows: Let x_0 an arbitrary fixed point of X,

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$W = W(x_0)$ be a path connected open neighborhood of x_0 and $\sigma_{x_0} = [\alpha]_{x_0}$ be a homotopy class of $(H_n)_{x_0}$. Since X is path connected, there exists a path γ with initial point x_0 and with terminal point x, for every $x \in W$. Therefore, the path γ determines an isomorphism $\gamma^*: (H_n)_{x_0} \to (H_n)_x$ defined by $\gamma^*\left([\alpha]_{x_0}\right) = [\beta]_x$ for any

$[\alpha]_{x_0} \in (H_n)_{x_0} \subset H_n$. Let us now define a mapping $s: W \to H_n$ such that $s(x) = \gamma^*\left([\alpha]_{x_0}\right) = [\beta]_x$ for every $x \in W$. If $c \in W$, then we define $s(c) = \gamma^*\left([\alpha]_c\right) = [\alpha]_c$, by taking $[\gamma] = [1] \in (H_n)_c$. It is seen that, the mapping s depends on both the homotopy classes $[\alpha]_{x_0}$ and $[\gamma]$. Suppose that the homotopy class $[\gamma]$ is chosen as arbitrary fixed, for each $x \in W$. So, the mapping s depends on only the homotopy class $[\alpha]_{x_0}$. s is well defined and $\Psi o s = 1_W$. Let us denote the totality of the mapping s defined over W by $\Gamma(W, H_n)$.

Let B be a basis of path connected open neighborhoods for each $x \in X$. Then,

$$T_n = \{s(W): W \in B,\ s \in \Gamma(W, H_n)\}$$

is a topology base on H_n [4]. In this topology, the mapping Ψ and s are continuous and Ψ is a local homomorphism. Thus, (H_n, Ψ) is a sheaf over X. (H_n, Ψ) (or only H_n) is called "The Sheaf H_n of Higher Homotopy Groups over X" [6,7]. For any open set W $\subset$ X, an element s of $\Gamma(W, H_n)$ is called a section of the sheaf H_n over W. The group $(H_n)_x = \pi_n(X, x)$ is called the stalk of the sheaf H_n for any $x \in X$. The set $\Gamma(W, H_n)$ is an abelian group with pointwise addition operation. Thus, the operation $+: H_n \oplus H_n \to H_n$ is continuous for every stalk of H_n [5]. Moreover, the group $(H_n)_x = \pi_n(X, x)$ is abelian for $n > 1$. Hence, H_n is a sheaf of abelian groups over X

The sheaf H_n satisfies the following properties:

1. Any two stalks of H_n are isomorphic with each other.

2. Let $W_1, W_2 \subset X$ be any open sets, $s_1 \in \Gamma(W_1, H_n)$ and $s_2 \in \Gamma(W_2, H_n)$. If $s_1(x_0) = s_2(x_0)$ for any point $x_0 \in W_1 \cap W_2$, then $s_1 = s_2$ over the whole $W_1 \cap W_2$ [10].

3. Let W $\subset$ X be an open set. Every section over W can be extended to a global section over X.

4. Let $x \in X$ be any point and W=W(x) be an open set. Then $\psi^{-1}(W) = \bigvee_{i \in I} s_i(W)$ and $\psi | s_i(W): s_i(W) \to W$ is a topological mapping for every $i \in I$. Hence, W=W(x) is evenly covered by ψ. Thus, (H_n, ψ) is an abelian covering space of X [9].

5. A topological stalk preserving mapping of H_n onto itself is called a sheaf isomorphism or a cover transformation, and the set of all cover transformation of H_n is denoted by T. Clearly, T is a group and isomorphic to the group $\Gamma(X, H_n)$. Hence, $(H_n)_x \cong \Gamma(X, H_n) \cong T$. Thus, T is transitive and H_n is a regular covering space of X [1].

Characterization

Let X_1, X_2 be any connected and locally path connected topological spaces and H_{n_1}, H_{n_2}

be the corresponding sheaves, respectively. Let us denote these as the pairs (X_1, H_{n_1}) and (X_2, H_{n_2}).

We begin by giving the following definitions.

Definition. 2.1. Let $f^*: H_{n_1} \to H_{n_2}$ be a mapping. If f^* is continuous, a homomorphism on each stalk of H_{n_1} and maps every stalk of H_{n_1} into stalk of H_{n_2}, then it is called a sheaf homomorphism.

Let $f: X_1 \to X_2$ be a continuous mapping and $f^*: H_{n_1} \to H_{n_2}$ be a sheaf homomorphism. If $f^*((H_{n_1})_{x_1}) \subset (H_{n_2})_{f(x_1)}$ for each $x_1 \in X_1$, then f* is called a stalk preserving homomorphism with respect to f $[2]$.

Definition 2.2. Let $f^*: H_{n_1} \to H_{n_2}$ be a sheaf homomorphism. If f* is homeomorphism then f* is called a sheaf isomorphism $[8]$.

Definition 2.3. Let the pairs (X_1, H_{n_1}) and (X_2, H_{n_2}) be given. If

1. The mapping $f: X_1 \to X_2$ is continuous,
2. The mapping $f^*: H_{n_1} \to H_{n_2}$ is continuous,
3. The mapping $f^*: H_{n_1} \to H_{n_2}$ is stalk preserving with respect to f.
4. The mapping $f^*\big|(H_{n_1})_{x_1}: (H_{n_1})_{x_1} \to (H_{n_2})_{f(x_1)}$ is a homomorphism for every $x_1 \in X_1$

then $(f, f^*): (X_1, H_{n_1}) \to (X_2, H_{n_2})$ is called a homomorphism between the pairs (X_1, H_{n_1}) and (X_2, H_{n_2}).

Definition 2.4. Let the pairs $\left(X_1, H_{n_1}\right)$, $\left(X_2, H_{n_2}\right)$ and the homomorphism $(f, f^*): \left(X_1, H_{n_1}\right) \to \left(X_2, H_{n_2}\right)$ be given. If the mappings f and f^* are homemorphisms, then (f, f^*) is called an isomorphism between the pairs (X_1, H_{n_1}) and (X_2, H_{n_2}) $[3]$.

Teorem 2.1. Let the pairs (X_1, H_{n_1}) and (X_2, H_{n_2}) be given. If the mapping $f^*: H_{n_1} \to H_{n_2}$ is given as a sheaf homomorphism, then there exists a unique continuous mapping $f: X_1 \to X_2$ such that the pair (f, f^*) is a homomorphism between the pairs (X_1, H_{n_1}) and (X_2, H_{n_2}).

Proof. To prove this teorem, we must first find a mapping $f: X_1 \to X_2$. However, for each $(H_{n_1})_{x_1} \subset H_{n_1}$ there exists a stalk $(H_{n_2})_{x_2} \subset H_{n_2} \ni f^*((H_{n_1})_{x_1}) \subset (H_{n_2})_{x_2}$, since f^* is stalk preserving. Therefore, to any point $x_1 \in X_1$ there uniquely corresponds a point $x_2 \in X_2$. If we denote this correspondence by $f(x_1) = x_2$,then we obtain a mapping $f: X_1 \to X_2$.

Let us now show that the mapping f is continuous. Let $W \subset f(X_1)$ be an open set. We may be prove that the set $f^{-1}(W)$ is an open set in X_1. Since W is an open set in X_2, there exists the arcwise connected open sets W_i in X_2, $i \in I$, such that $W = \bigcup_{i \in I} W_i$. Thus

$$s^2(W) = \bigcup_{i \in I} s_i^2(W_i)$$

is an open set in H_{n_2}, for a section $s^2 \in \Gamma(W, H_{n_2})$. However,

$$f^{*^{-1}}\left(s^2(W)\right)=\bigcup_{i\in I} f^{*^{-1}}\left(s_i^2(W_i)\right)$$

is an open set in H_{n_1}, since f^* is continuous. Thus, there exists the arcwise connected open sets V_i in X_1, $i\in I$, such that

$$f^{*^{-1}}\left(s^2(W)\right)=\bigcup_{i\in I} s_i^1(V_i)$$

where s_i^1 ' s are section over V_i for each $i\in I$. Hence

$$\psi_1\left(f^{*^{-1}}\left(s^2(W)\right)\right)=\bigcup_{i\in I} V_i$$

is an open set in X_1. Let us now show that

$$f^{-1}(W)=\bigcup_{i\in I} V_i.$$

1. Let $x_1\in f^{-1}(W)$. Then, there exists only one point $x_2\in X_2 \ni f(x_1)=x_2$. Hence $s^2(x_2)=\sigma_{x_2}\in s^2(W)$ and there is an element $\sigma_{x_1}\in f^{*^{-1}}\left(s^2(W)\right)\ni f^*(\sigma_{x_1})=\sigma_{x_2}$, $\sigma_{x_1}\in s_i^1(V_i)$, for an $i\in I$, since $f^{*^{-1}}\left(s^2(W)\right)=\bigcup_{i\in I} s_i^1(V_i)$. Hence $\psi_1(\sigma_{x_1})=x_1\in V_i$. Therefore $f^{-1}(W)\subset\bigcup_{i\in I} V_i$.

2. Let $x_1\in\bigcup_{i\in I} V_i$. Then $x_1\in V_i$ and $s_i^1(x_1)\in(H_{n_1})_{x_1}$, for an $i\in I$. Therefore $f^*\left(s_i^1(x_1)\right)\in s^2(W)$ and $\psi_2\left(f^*\left(s_i^1(x_1)\right)\right)=x_2\in W.$

From the definition of f, $f(x_1)=x_2$. Thus $x_1\in f^{-1}(W)$. Also,

$$\bigcup_{i\in I} V_i\subset f^{-1}(W).$$

Thus the mapping $f: X_1\to X_2$ is continuous. On the other hand it can be shown that the pair (f,f^*) is a homomorphism between the pairs (X_1,H_{n_1}) and (X_2,H_{n_2}), and f is unique, since $f\circ\psi_1=\psi_2\circ f^*$.

We can now state the following theorem.

Theorem 2.2. Let the pairs (X_1,H_{n_1}), (X_2,H_{n_2}) and (X_3,H_{n_3}) be given. If the mappings $f_1^*: H_{n_1}\to H_{n_2}$ and $f_2^*: H_{n_2}\to H_{n_3}$ are sheaf homomorphisms, then there exists a homomorphism between the pairs (X_1,H_{n_1}) and (X_3,H_{n_3}) such that $f=f_2\circ f_1$, $f^*=f_2^*\circ f_1^*$.

Proof. Since the mappings f_1^*, f_2^* are continuous, the mapping $f_2^*\circ f_1^*$ is also continuous. By Theorem 2.1, there is a continuous mapping f from X_1 into X_3. Clearly $f_2^*\circ f_1^*$ preserves the stalk with respect to f and $f_2^*\circ f_1^*$ is a homomorphism on each stalk. Hence the pair $(f, f_2^*\circ f_1^*)$ is a homomorphism between the pairs (X_1,H_{n_1}) and (X_3,H_{n_3}). Now, let us show that $f=f_2\circ f_1$. Since, for any stalk $(H_{n_1})_{x_1}\subset H_{n_1}$, there is a stalk $(H_{n_2})_{x_2}\subset H_{n_2}\ni f_1^*((H_{n_1})_{x_1})\subset(H_{n_2})_{x_2}$ and for any stalk $(H_{n_2})_{x_2}\subset H_{n_2}$, there is a stalk $(H_{n_3})_{x_3}\subset H_{n_3}\ni f_2^*((H_{n_2})_{x_2})\subset(H_{n_3})_{x_3}$,

$$(f_2^*\circ f_1^*)((H_{n_1})_{x_1})=f_2^*(f_1^*(H_{n_1})_{x_1})\subset f_2^*((H_{n_2})_{x_2})\subset(H_{n_3})_{x_3},$$

and $f(x_1)=x_3$. On the other hand $f_1(x_1)=x_2$ and $f_2(x_2)=x_3$ since

$f_1^*((H_{n_1})_{x_1}) \subset (H_{n_2})_{x_2}$, $f_2^*((H_{n_2})_{x_2}) \subset (H_{n_3})_{x_3}$.

So, $(f_2 \, of_1)(x_1) = f_2(f_1(x_1)) = x_3$. Therefore $f_2 \, of_1 = f$.

Now, we can give the following theorem.

Theorem 2.3. Let the pairs (X_1, H_{n_1}) and (X_2, H_{n_2}) be given. If the mapping $f^*: H_{n_1} \to H_{n_2}$ is a sheaf isomorphism, then there exists an isomorphism between the pairs (X_1, H_{n_1}) and (X_2, H_{n_2}).

Proof. It follows from the theorem 2.1 that, there exists a continuous mapping $f: X_1 \to X_2$. Let us now show that f is a bijection. In fact, for any two elements $x_1, y_1 \in X_1$, if $f(x_1) = f(y_1) = x_2$, then there is a stalk $(H_{n_2})_{x_2} \subset H_{n_2}$, $x_2 \in X_2 \ni$ $f^*((H_{n_1})_{x_1}) = f^*((H_{n_1})_{y_1}) = (H_{n_2})_{x_2}$. However, this is impossible, since f^* is one-to-one. Therefore $x_1 = y_1$. On the other hand, for each stalk $(H_{n_2})_{x_2}$, there exists a stalk $(H_{n_1})_{x_1} \ni f^*((H_{n_1})_{x_1}) = (H_{n_2})_{x_2}$ since f^* is onto. It follows from this reason that, for each $x_2 \in X_2$, there exists an element $x_1 \in X_1 \ni f(x_1) = x_2$. Hence f is a bijection. By Theorem 2.1, there is a continuous mapping $g: X_2 \to X_1$, since $f^{*^{-1}}$ is continuous. It is similarly shown that g is a bijection. On the other hand, it can be shown that $g = f^{-1}$ Therefore f is a homeomorphism.

Clearly, $f^{*^{-1}}$ preserves the stalk with respect to f. Thus the pair (f, f^*) is an isomorphism.

Now, let C be the category of the sheaves of higher homotopy groups and sheaf homomorphisms and D be the category of the connected and locally path connected topological spaces and continuous mappings. Then, we can define a mapping

$F: C \to D$ as follows:

For any sheaf H_n and every morphism $f^*: H_{n_1} \to H_{n_2}$, let $F(H_n) = X$ and $F(f^*) = f: X_1 \to X_2$. Then,

1. If $f^* = 1_{H_{n_1}}$, then $F(1_{H_{n_1}}) = 1_{X_1}$.
2. If $f_1^*: H_{n_1} \to H_{n_2}$ and $f_2^*: H_{n_2} \to H_{n_3}$ are two sheaf homomorphisms, then $F(f_2^* \, of_1^*) = F(f_2^*) \, oF(f_1^*)$.

Thus, we can state the following theorem.

Theorem 2.4. There is a covariant functor from the category of the sheaves of higher homotopy groups and sheaf homomorphisms to the category of the connected and locally path connected topological spaces and continuous mappings.

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