Complex Elements in R^4

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Abstract: *In this paper, the concepts complex elements and minimal elements are extended to the real space R^4 which are defined for the real space R^3. The properties of these elements are presented.*

Key Words: **Complex elements, minimal elements, minimal flat, minimal hypercone.**

R^4 de Sanal Elemanlar

Özet: *Bu çalışmada, R^3 reel uzayı için tanımlanan sanal elemanlar ve minimal elemanlar kavramları, R^4 reel uzayına genişletilmiş ve bu elemanların özellikleri verilmiştir.*

Anahtar Kelimeler: **Sanal elemanlar, minimal elemanlar, minimal flat, minimal hiperkoni**

Introduction

Complex elements in the real space R^3 were defined, also the necessary and the sufficient conditions were presented for a complex element to be a minimal element by Şemin in [1]. Complex elements and minimal elements are some important tools of R^3 to investigate the properties of the cone $x^2+y^2+z^2=0$, which has no real points except the point $O(0,0,0)$. The complex elements in R^3 are the complex points, lines, and planes of R^3.

In this paper, the concepts complex elements and minimal elements are extended to the real space R^4 which are defined for the real space R^3. The complex elements and minimal elements are some important tools of R^4 to examine the hypercone $x^2+y^2+z^2+v^2=0$, which has no real points except the point $O(0,0,0,0)$. Throughout this paper, we mean the complex points, lines, planes, and flats of R^4 with the complex elements in R^4. Forsyth defined the real flats in R^4 and presented the properties of the real elements of R^4 in [2].

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Now we define the complex elements in R^4. A point $N(x_1,x_2,x_3,x_4)$ is called a complex point in R^4 if any component of the point is a complex number. Any two complex points, the components of which are respectively conjugate to each other, are said to be conjugate complex points. Let $N(x_1,x_2,x_3,x_4)$ be a complex point in R^4. The complex numbers $X_1;X_2;X_3;X_4;X_5$ are said to be homogeneous coordinates of the complex point if there is a relation $x_p = X_p/X_5$ for $p=1,2,3,4$. Thus a complex point in R^4 is also denoted by $N(X_1,X_2,X_3,X_4,X_5)$ in the homogeneous coordinates. In addition to this, a complex point at infinity is denoted by $N(X_1,X_2,X_3,X_4,0)$.

Definition 1. The equation $\Delta=\sum_{p=1}^{5} A_p X_p = 0$ defines a complex flat in R^4 if any coefficient A_p is a complex number and the coefficients A_p for $p=1,2,3,4,5$ are not proportional.

Definition 2. Let two complex flats be given by the $\Delta=\sum_{p=1}^{5} A_p X_p = 0$ and $\overline{\Delta}=\sum_{p=1}^{5} \overline{A}_p X_p = 0$. The complex flats are said to be conjugate flats if A_p and $\overline{A}_p$ are respectively conjugate to each other for $p=1,2,3,4,5$.

In the lights of the Definitions 1 and 2, a complex plane and a complex line are respectively considered as an intersection of two non-parallel complex flats and an intersection of three non-parallel complex flats.

Some Properties of Complex Elements in R^4

Now we give the main theorems and corollaries about the complex elements in the real space R^4.

Theorem 1. The real points satisfying a complex flat equation are on a real plane which has a finite or an infinite distance from the origin.

Proof: Let a complex flat be given by the equation $\Delta=\sum_{p=1}^{5} A_p X_p = 0$. The flat equation is also written in the form $\Delta=\Delta_1+i\Delta_2=0$, where $\Delta_1=0$ and $\Delta_2=0$ are real flats in R^4. If the real flats $\Delta_1=0$ and $\Delta_2=0$ are non-parallel to each other the intersection of them is a real plane which has a finite distance from the origin. For if the real flats are parallel to each other then the plane is at infinity. The theorem is proved. ■

Corollary 1. A complex flat has only one real plane.

Proof: The solutions of the homogenous equation system consisting of the equations $\Delta_1=0$ and $\Delta_2=0$, which is given in Theorem 1, state a unique plane in R^4. The proof is complete. ■

Corollary 2. The complex flat $\Delta=0$ passes through a real plane which is the unique real plane of the conjugate complex flat of $\Delta=0$.

Proof: The proof is obvious from Definition 2 and Theorem 1. ■

Corollary 3. There is only one real line on a complex plane in R^4.

Proof: Let a complex plane be given by $\Delta = 0$ and $\Delta' = 0$. It has been given that there is only one real plane on a complex flat so the intersection of the planes, which are determined by the complex flats $\Delta = 0$ and $\Delta' = 0$, determines only one line if the planes are non-parallel to each other. The proof is complete. ■

Corollary 4. There is only one real point on a complex line.

Proof: Let a complex line be determined by the complex flats $\Delta = 0$, $\Delta' = 0$, and $\Delta'' = 0$ in which any two of them are not parallel to each other. The real planes, which are on the complex flats, meet in a real point because the complex flats are non-parallel to each other. The proof is complete. ■

Minimal Elements in R^4

In this section the minimal elements are defined for R^4. It is presented that the necessary and the sufficient condition for a complex flat to be a minimal flat.

Definition 3. The complex points having the square distances from the origin is zero define a hypercone which is called a minimal hypercone with the vertex origin and is given by the equation $\sum_{p=1}^{4} x_p^2 = 0$.

The minimal hypercone is also given by the equation $\sum_{p=1}^{4} \left(X_p - a_p X_5\right)^2 = 0$ in the homogeneous coordinates. The intersection of the minimal hypercone and a flat at infinity is a quadratic curve given by the equation

$$\sum_{p=1}^{4} X_p^2 = 0, \quad X_5 = 0. \tag{1}$$

The equation (1) defines all minimal hypercones in R^4. In [3] Woods gave the following definition.

Definition 4. The curve which is given by the equation (1) is said to be an absolute of the real flat at infinity.

Definition 5. A direction is a minimal direction in R^4 if the point of the direction at infinity is on the absolute.

Lemma 1. A direction (b_1, b_2, b_3, b_4) is a minimal direction if and only if $\sum_{p=1}^{4} b_p^2 = 0$.

Proof: The proof is obvious from Definition 6. ■

Definition 6. A vector which has a minimal direction is called a minimal vector.

Lemma 2. A vector $\vec{v}$ is a minimal vector if and only if $\vec{v}^2 = 0$.

Proof: From Definition 7 if $\vec{v}$ is a minimal vector then $\vec{v}$ has a minimal direction. By Lemma 1 it is obvious that $\vec{v}^2 = 0$. Conversely if $\vec{v}^2 = 0$ then $\vec{v}$ has a minimal direction so the vector is a minimal vector. ■

Definition 7. A complex line is said to be a minimal line if the directrix of the complex line is a minimal vector.

Theorem 2. A complex line is a minimal line if and only if the distance between any two points of the complex line is zero.

Proof: Let $A(a_1,a_2,a_3,a_4)$ be a complex point and let $\vec{v}(v_1,v_2,v_3,v_4)$ be a minimal vector. From Definition 7 the complex line given by the equation

$$\frac{x_1-a_1}{v_1}=\frac{x_2-a_2}{v_2}=\frac{x_3-a_3}{v_3}=\frac{x_4-a_4}{v_4}=t,$$

where t is a nonzero complex variable, is a minimal line. Let $P(x_1,x_2,x_3,x_4)$ be an arbitrary point on the complex line. If the distance between $A(a_1,a_2,a_3,a_4)$ and $P(x_1,x_2,x_3,x_4)$ is denoted by $|AP|$ then

$$|AP|=\left(\sum_{p=1}^{4}(x_p-a_p)^2\right)^{\frac{1}{2}}=\left(\sum_{p=1}^{4}(v_pt+a_p-a_p)^2\right)^{\frac{1}{2}}=\left(t^2.\sum_{p=1}^{4}v_p^2\right)^{\frac{1}{2}}=0$$

since t is nonzero and $\vec{v}(v_1,v_2,v_3,v_4)$ is a minimal vector. Conversely let $A(a_1,a_2,a_3,a_4)$ and $B(b_1,b_2,b_3,b_4)$ be two complex points on the complex line and let $|AB|=\left(\sum_{p=1}^{4}(a_p-b_p)^2\right)^{\frac{1}{2}}=0$. Then the equation of the complex line can be written as

$$\frac{x_1-a_1}{a_1-b_1}=\frac{x_2-a_2}{a_2-b_2}=\frac{x_3-a_3}{a_3-b_3}=\frac{x_4-a_4}{a_4-b_4}=t.$$

Thus the directrix of the complex line is the vector $\vec{v}(a_1-b_1,a_2-b_2,a_3-b_3,a_4-b_4)$. The vector $\vec{v}$ is a minimal vector since $|AB|=0$. Thus the theorem is proved. ■

Definition 8. A complex flat in R^4 is said to be a minimal flat if the normal vector of the complex flat is a minimal vector.

From Definition 8 and 9 it is clear that the intersection of two non-parallel minimal flats states a minimal plane and the normal vectors of the minimal plane are the normal vectors of the minimal flats.

Corollary 5. A minimal flat is tangent to the absolute along the normal vector of the minimal flat.

Proof: Let a minimal flat be given by the equation $A_1x_1+A_2x_2+A_3x_3+A_4x_4+A_5=0$. It is clear that $\sum_{p=1}^{4}A_p^2=0$ since $\vec{v}(A_1,A_2,A_3,A_4)$, which is the normal vector of the minimal flat, is a minimal vector. Thus the vector $\vec{v}$ is on the minimal hypercone $x_1^2+x_2^2+x_3^2+x_4^2=0$. Then the minimal flat is tangent to the absolute along the vector $\vec{v}(A_1,A_2,A_3,A_4)$ since the absolute states the all minimal hypercones in R^4. The proof is complete. ■

As a consequence of Corollary 5, it is seen that a minimal plane is tangent to the absolute along two normal vectors of the minimal plane since an intersection of two minimal flats states a minimal plane from the definition of the complex planes.

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