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Poisson Integral Formula For An Elliptic Type Equation

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Abstract : *In this study, Poisson integral formulas are obtained for the GASPT equation and for a partial differential equation of elliptic type defined in a domain with the boundary given by a non-isotropic distance.*

Key Words: **GASPT equation, Poisson integral formula, Dirichlet problem**

Eliptik Tipten Bir Denklem İçin Poisson İntegral Formülü

Özet: *Bu çalışmada, GASPT denklemi ve sınırı non-izotropik uzaklıkla verilen bir bölgede tanımlı, değişken katsayılı eliptik tipten bir denklemin çözümleri için Poisson integral formülü elde edilmiştir.*

Anahtar Kelimeler: **GASPT denklemi, Poisson integral formülü, Dirichlet problemi**

1. Introduction

Let us consider the open disc

$$D = \{(x,y):\ x^2 + y^2 < a^2\}$$

and its boundary

$$\partial D = \{(x,y):\ x^2 + y^2 = a^2\}$$

defined in xy-plane. Here a is a positive constant. The problem

$$\begin{aligned} \Delta u = u_{xx} + u_{yy} = 0 \qquad & in\ \ D \\ u(x,y) = f(x,y) \qquad & on\ \ \partial D \end{aligned} \tag{1}$$

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defines a Dirichlet problem for a circle. Here, Δ shows as usual the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

By making the transformations $x = r\cos\theta$, $y = r\sin\theta$, we can redefine the problem in polar coordinates as

$$\begin{aligned} &\Delta u^* = u^*_{rr} + \frac{1}{r} u^*_r + \frac{1}{r^2} u^*_{\theta\theta} = 0 \quad ; \quad 0 < r < a \\ &u^*(a,\theta) = f(\theta) \qquad\qquad ; \quad 0 \le \theta \le 2\pi \end{aligned} \tag{2}$$

The solution of problem (2) is given by the well- known Poisson integral formula [2,3].

$$u^*(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \tau) + r^2} f(\tau) d\tau \tag{3}$$

2. Poisson Integral Formulas

Now let us consider the well-known GASPT (Generally Auxiliary Symmetric Potential Theory) operator [1,4]

$$L = \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\alpha_i}{x_i} \frac{\partial}{\partial x_i} \right) + \frac{\partial^2}{\partial y^2} \tag{4}$$

where α_i $(i = 1,2,\ldots,n)$ is constant, and the elliptic type operator

$$L_1 = \sum_{i=1}^{2} \left(\frac{1}{m_i^2} x_i^{2-2m_i} \frac{\partial^2}{\partial x_i^2} - \frac{1}{m_i} \left(1 - \frac{1}{m_i} \right) x_i^{1-2m_i} \frac{\partial}{\partial x_i} \right) \tag{5}$$

and let

$$\begin{aligned} D_1 &= \{(x_1,\ldots,x_n,y) : x_1^2 + \ldots + x_n^2 + y^2 < a^2\} \\ \partial D_1 &= \{(x_1,\ldots,x_n,y) : x_1^2 + \ldots + x_n^2 + y^2 = a^2\} \\ D_2 &= \{(x_1,x_2) : x_1^{2m_1} + x_2^{2m_2} < a^2\} \\ \partial D_2 &= \{(x_1,x_2) : x_1^{2m_1} + x_2^{2m_2} = a^2\} \end{aligned}$$

where a is a positive constant, m_1 and m_2 are positive integers. In [1], some particular solutions of GASPT equation defined by the operator (1) is obtained and in [4], Harnack type inequalities for the positive solutions of the same equation are given.

Let us consider the following two Dirichlet problems:

$$\begin{aligned} Lu(x_1,x_2,...,x_n,y)=0 \quad &, \quad in\ D_1 \\ u(x_1,x_2,...,x_n,y)=f(\sqrt{x_1^2+x_2^2+...+x_n^2}\ ,y), &\quad on\ \partial D_1 \end{aligned} \tag{6}$$

and

$$\begin{aligned} L_1u(x_1,x_2)=0 \quad &, \quad in\ D_2 \\ u(x_1,x_2)=f(x_1,x_2)\ , &\quad on\ \partial D_2 \end{aligned} \tag{7}$$

In this study, we give the solutions of these two problems in terms of an integral which includes the boundary values and a Poisson type kernel. Thus we refer the solutions as Poisson integral formulas for the Dirichlet problems. Let us first give the following lemma.

Lemma 1. Let

$$n-1+\sum_{i=1}^{n}\alpha_i=0. \tag{8}$$

Then, under the transformation

$$x^2=x_1^2+x_2^2+...+x_n^2, \tag{9}$$

the equation

$$Lu=\sum_{i=1}^{n}\left(\frac{\partial^2 u}{\partial x_i^2}+\frac{\alpha_i}{x_i}\frac{\partial u}{\partial x_i}\right)+\frac{\partial^2 u}{\partial y^2}=0 \tag{10}$$

is reduced to the Laplace equation.

$$Lu_1^*=\frac{\partial^2 u_1^*}{\partial x^2}+\frac{\partial^2 u_1^*}{\partial y^2}=0 \tag{11}$$

Proof. Since $x^2=x_1^2+x_2^2+...+x_n^2$ and $u(x_1,...,x_n,y)=u_1^*\left(\sqrt{x_1^2+...+x_n^2}\ ,\ y\right)$

we have

$$\frac{\partial x}{\partial x_i}=\frac{x_i}{x},$$

$$\frac{\partial^2 x}{\partial x_i^2}=\frac{x^2-x_i^2}{x^3}.$$

Thus by the chain rule

$$\frac{\partial u}{\partial x_i}=\frac{x_i}{x}\frac{\partial u_1^*}{\partial x_i},$$

and

$$\frac{\partial^2 u}{\partial x_i^2}=\left(\frac{x_i}{x}\right)^2\frac{\partial^2 u_1^*}{\partial x^2}+\frac{\left(x^2-x_i^2\right)}{x^3}\frac{\partial u_1^*}{\partial x}.$$

By substituting these values in (10), we obtain

$$Lu_1^* = \frac{\partial^2 u_1^*}{\partial x^2} + \frac{n-1+\sum_{i=1}^{n}\alpha_i}{x}\frac{\partial u_1^*}{\partial x} + \frac{\partial^2 u_1^*}{\partial y^2} = 0.$$

Since $n-1+\sum_{i=1}^{n}\alpha_i = 0$, we get the Laplace equation

$$Lu_1^* = \frac{\partial^2 u_1^*}{\partial x^2} + \frac{\partial^2 u_1^*}{\partial y^2} = 0,$$

which gives the proof.

Theorem 1. The solution of the Dirichlet problem defined by (6) under the condition (8) is given by the Poisson integral formula

$$u^{**}(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{a^2-\rho^2}{a^2-2a\rho\cos(\theta-\tau)+\rho^2}f(\tau)d\tau$$

where $\rho = \sqrt{x_1^2+\ldots+x_n^2+y^2}$ and $\theta = \arctan\frac{y}{\sqrt{x_1^2+\ldots+x_n^2}}$

Proof. By Lemma 1, under the condition (8), the equation (10) can be reduced to the Laplace equation

$$Lu_1^* = \frac{\partial^2 u_1^*}{\partial x^2} + \frac{\partial^2 u_1^*}{\partial y^2} = 0$$

and hence with the transformation (9), Dirichlet problem (6) is transformed to the problem

$$Lu(x_1,\ldots,x_n,y) = Lu_1^*\left(\sqrt{x_1^2+\ldots+x_n^2}, y\right) = 0 \quad ; \; in \; D$$
$$u(x_1,\ldots,x_n,y) = u_1^*\left(\sqrt{x_1^2+\ldots+x_n^2}, y\right) = f(x,y) \quad ; \; on \; \partial D$$

We know that the solution of this problem, in polar coordinates is given by the Poisson integral formula

$$u^{**}(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{a^2-\rho^2}{a^2-2a\rho\cos(\theta-\tau)+\rho^2}f(\tau)d\tau.$$

Thus, the function

$$u(x_1,\ldots,x_n,y) = u_1^*(x,y) = u^{**}\left(\sqrt{x_1^2+\ldots+x_n^2+y^2}\;,\; arctan\frac{y}{\sqrt{x_1^2+\ldots+x_n^2}}\right)$$

gives the solution of problem (6).

Now, we obtain the Poisson integral formula for the problem (7).

Lemma 2. The equation

$$L_1 u = \sum_{i=1}^{2}\left(\frac{1}{m_i^2}x_i^{2-2m_i}\frac{\partial^2 u}{\partial x_i^2}-\frac{1}{m_i}\left(1-\frac{1}{m_i}\right)x_i^{1-2m_i}\frac{\partial u}{\partial x_i}\right)=0 \tag{12}$$

under the change of variables

$$x_1=(r_1\cos\theta)^{\frac{1}{m_1}} \quad , \quad x_2=(r_1\sin\theta)^{\frac{1}{m_2}}, \tag{13}$$

is transformed to the Laplace equation in polar coordinates

$$L_1 u_1 = \frac{\partial^2 u_1}{\partial r_1^2}+\frac{1}{r_1^2}\frac{\partial^2 u_1}{\partial \theta^2}+\frac{1}{r_1}\frac{\partial u_1}{\partial r_1}=0 \tag{14}$$

Proof. Since $x_1=(r_1\cos\theta)^{1/m_1}$,and $x_2=(r_1\sin\theta)^{1/m_2}$ we have

$$r_1^2=x_1^{2m_1}+x_2^{2m_2}, \qquad \theta=\arctan\left(x_1^{-m_1}x_2^{m_2}\right). \tag{15}$$

In addition,

$$\frac{\partial u}{\partial x_1}=m_1(x_1^{2m_1}+x_2^{2m_2})^{\frac{-1}{2}}x_1^{2m_1-1}\frac{\partial u_1}{\partial r_1}-m_1x_1^{m_1-1}x_2^{m_2}(x_1^{2m_1}+x_2^{2m_2})^{-1}\frac{\partial u_1}{\partial \theta}$$

$$\frac{\partial u}{\partial x_2}=m_2(x_1^{2m_1}+x_2^{2m_2})^{\frac{-1}{2}}x_2^{2m_2-1}\frac{\partial u_1}{\partial r_1}-m_2x_1^{m_1}x_2^{m_2-1}(x_1^{2m_1}+x_2^{2m_2})^{-1}\frac{\partial u_1}{\partial \theta}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x_1^2}&=m_1^2(x_1^{2m_1}+x_2^{2m_2})^{-1}x_1^{4m_1-2}\frac{\partial^2 u_1}{\partial r_1^2}-2m_1^2(x_1^{2m_1}+x_2^{2m_2})^{\frac{-3}{2}}x_1^{3m_1-2}x_2^{m_2}\frac{\partial^2 u_1}{\partial r_1\partial\theta}\\&+m_1^2x_1^{2m_1-2}x_2^{2m_2}(x_1^{2m_1}+x_2^{2m_2})^{-2}\frac{\partial^2 u_1}{\partial \theta^2}\\&+\left[m_1(2m_1-1)(x_1^{2m_1}+x_2^{2m_2})^{\frac{-1}{2}}x_1^{2m_1-2}-m_1^2x_1^{4m_1-2}(x_1^{2m_1}+x_2^{2m_2})^{\frac{-3}{2}}\right]\frac{\partial u_1}{\partial r_1}\\&+\left[-m_1(m_1-1)x_1^{m_1-2}x_2^{m_2}(x_1^{2m_1}+x_2^{2m_2})^{-1}+2m_1^2x_2^{m_2}x_1^{3m_1-2}(x_1^{2m_1}+x_2^{2m_2})^{-2}\right]\frac{\partial u_1}{\partial \theta}\end{aligned}$$

$$\frac{\partial^2 u}{\partial x_2^2} = m_2^2(x_1^{2m_1}+x_2^{2m_2})^{-1}x_2^{4m_1-2}\frac{\partial^2 u_1}{\partial r_1^{\,2}} - 2m_2^2(x_1^{2m_1}+x_2^{2m_2})^{\frac{-3}{2}}x_1^{m_1}x_2^{3m_2-2}\frac{\partial^2 u_1}{\partial r_1\partial\theta}$$

$$+m_2^2x_1^{2m_1}x_2^{2m_2-2}(x_1^{2m_1}+x_2^{2m_2})^{-2}\frac{\partial^2 u_1}{\partial\theta^2}$$

$$+\left[m_2(2m_2-1)(x_1^{2m_1}+x_2^{2m_2})^{\frac{-1}{2}}x_2^{2m_1-2} - m_2^2x_2^{4m_1-2}(x_1^{2m_1}+x_2^{2m_2})^{\frac{-3}{2}}\right]\frac{\partial u_1}{\partial r_1}$$

$$+\left[m_2(m_2-1)x_1^{m_1}x_2^{m_2-2}(x_1^{2m_1}+x_2^{2m_2})^{-1} - 2m_2^2x_1^{m_2}x_2^{3m_2-2}(x_1^{2m_1}+x_2^{2m_2})^{-2}\right]\frac{\partial u_1}{\partial\theta}$$

Substituting these values in (12) we obtain (14).

Theorem 2. The solution of the problem defined by (7) is

$$u_1(r_1,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{a^2-r_1^{\,2}}{a^2-2ar_1\cos(\theta-\tau)+r_1^{\,2}}f_1(\tau)d\tau$$

where $r_1^2 = x_1^{2m_1}+x_2^{2m_2}$, $\quad \theta = \arctan\left(x_1^{-m_1}x_2^{m_2}\right)$.

Proof. By Lemma 2, we have

$$L_1u_1 = \frac{\partial^2 u_1}{\partial r_1^{\,2}}+\frac{1}{r_1^{\,2}}\frac{\partial^2 u_1}{\partial\theta^2}+\frac{1}{r_1}\frac{\partial u_1}{\partial r_1}=0$$

and by (13), the boundary condition is

$$u\left((a\cos\theta)^{\frac{1}{m_1}}\ ,\ (a\sin\theta)^{\frac{1}{m_2}}\right)=f\left((a\cos\theta)^{\frac{1}{m_1}},\ (a\sin\theta)^{\frac{1}{m_2}}\right)$$

$$u_1(a,\theta)=f_1(\theta) \qquad ; \quad 0\le\theta\le 2\pi$$

Hence by (2), the solution of this problem is given by Poisson integral formula

$$u_1(r_1,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{a^2-r_1^{\,2}}{a^2-2ar_1\cos(\theta-\tau)+r_1^{\,2}}f_1(\tau)d\tau .$$

Thus

$$u(x_1,x_2)=u_1\left(\left(x_1^{2m_1}+x_2^{2m_2}\right)\ ,\ \arctan\left(x_1^{-m_1}x_2^{m_2}\right)\right)$$

gives the solution of problem (7).

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