on The Hadamard Products of Its Adjoint Matrix With a Square Matrix

Dursun TAŞCI

Abstract: In this paper for any matrix \( A = (a_{ij}) \in M_n(R) \) we defined \( \varphi(A) = A \circ \text{adj}A \) and \( \varphi_T(A) = A \circ (\text{adj}A)^T \), where \( T \) denotes transpose and \( \circ \) denotes Hadamard product. We obtained some properties of \( \varphi(A) \) and \( \varphi_T(A) \).

Key Words: Hadamard product, Adjoint matrix

Bir Kare Matris ile Onun Adjoint Matrisinin Hadamard Çarpımları Üzerine

Özet: Bu çalışmadada reel elemanlı herhangi bir \( n \times n \) A kare matrisi göz önüne alınarak \( \varphi(A) = A \circ \text{adj}A \) ve \( \varphi_T(A) = A \circ (\text{adj}A)^T \) matrisleri tanımlandı ve bu matrislerin bazı özellikleri elde edildi.

Anahtar Kelimeler: Hadamard Çarpımı, Adjoint Matris.

Introduction and the Main Results

Firstly we give the following definitions

Definition 1.[1] The Hadamard product of \( A = (a_{ij}) \in M_n \) and \( B = (b_{ij}) \in M_n \) is defined by \( A \circ B = (a_{ij}b_{ij}) \in M_n \).

Definition 2.[2] Let \( A = (a_{ij}) \) be an \( n \times n \) matrix over any commutative ring. The permanent of \( A \), written \( \text{per}(A) \), is defined by

\[
\text{per}(A) = \sum_{\sigma} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)},
\]

where the summation extends over all one-to-one functions from \( \{1, 2, \ldots, n\} \) to \( \{1, 2, \ldots, n\} \).

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1 Selcuk University, Department of Mathematics, [42031]Campus/Konya/TURKEY
Definition 3.[3] For any matrix $A = (a_{ij}) \in M_n$, the column matrix norm $\|A\|_1$ is defined by

$$
\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|.
$$

Definition 4.[3] For any matrix $A = (a_{ij}) \in M_n$, the row matrix norm $\|A\|_\infty$ is defined by

$$
\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.
$$

Definition 5.[3] For any $A = (a_{ij}) \in M_n$, the Euclidean matrix norm is defined by

$$
\|A\|_E = \left( \sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}.
$$

Definition 6.[3] For any matrix $A = (a_{ij}) \in M_n$, sum matrix norm $\|A\|_t$ is defined by

$$
\|A\|_t = \sum_{i,j=1}^{n} |a_{ij}|.
$$

Now we present the main results.

Theorem 1. For any $A = (a_{ij}) \in M_2(\mathbb{R})$

$$
\det(\varphi(A)) = \det(A)\text{per}(A)
$$

and

$$
\det(\varphi_T(A)) = \det(A)\text{per}(A),
$$

where $M_2(\mathbb{R})$ denotes $2 \times 2$ matrices with real entries, $\varphi(A) = A \circ \text{adj}A$ and $\varphi_T(A) = A \circ (\text{adj}A)^T$.

Proof. For any matrix

$$
A = \begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix}
$$

since

$$
\text{adj}A = \begin{bmatrix}
    a_{22} & -a_{12} \\
    -a_{21} & a_{11}
\end{bmatrix}
$$

we write

$$
\varphi(A) = A \circ \text{adj}A = \begin{bmatrix}
    a_{11}a_{22} & -(a_{12})^2 \\
    -(a_{21})^2 & a_{11}a_{22}
\end{bmatrix}.
$$
If we compute the determinant of $\varphi(A)$, then we find
\[
\det(\varphi(A)) = (a_{11}a_{22})^2 - (a_{12}a_{21})^2
= (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{22} + a_{12}a_{21})
= \det(A)\text{per}(A).
\]
Similarly it is easily seen that
\[
\det(\varphi_T(A)) = \det(A)\text{per}(A).
\]
Thus the proof is complete.

**Remark 1.** Unfortunately Theorem 1 is wrong for $n \geq 3$. Let us give a counterexample. Consider
\[
A = \begin{bmatrix}
1 & 2 & 3 \\
1 & 2 & 1 \\
2 & 2 & 4
\end{bmatrix}.
\]
Since
\[
\text{adj } A = \begin{bmatrix}
6 & -2 & -4 \\
-2 & -2 & 2 \\
-2 & 2 & 0
\end{bmatrix}
\]
we find
\[
\varphi(A) = \begin{bmatrix}
6 & -4 & -12 \\
-2 & -4 & 2 \\
-4 & 4 & 0
\end{bmatrix}.
\]
On the other hand, since $\det(\varphi(A)) = 272$, $\det(\varphi_T(A)) = 216$, $\det(A) = -4$ and $\text{per}A = 40$, it follows that
\[
\det(\varphi(A)) \neq \det(A)\text{per}(A)
\]
and
\[
\det(\varphi_T(A)) \neq \det(A)\text{per}(A).
\]

**Theorem 2.** For any matrix $A = (a_{ij}) \in M_2(\mathbb{R})$, the eigenvalues of $\varphi(A)$ are $\lambda_1 = \det A$, $\lambda_2 = \text{per}(A)$. Also the eigenvalues of $\varphi_T(A)$ are $\lambda_1 = \det A$, $\lambda_2 = \text{per}(A)$.

**Proof.** For any matrix
\[
A = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\]
since
\[
\varphi(A) = \begin{bmatrix}
a_{11}a_{22} & -(a_{12})^2 \\
-(a_{21})^2 & a_{11}a_{22}
\end{bmatrix}
\]
we have
\[
\det(\lambda I - \varphi(A)) = \det \begin{bmatrix}
\lambda - a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & \lambda - a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & \lambda - a_{nn}
\end{bmatrix} (1.1)
\]

\[
= \lambda^2 - 2a_{11}a_{22}\lambda + (a_{11}a_{22} - a_{12}a_{21})^2
\]
On the other hand the roots of the equation (1.1) are
\[
\lambda_1 = a_{11}a_{22} - a_{12}a_{21} \quad \text{and} \quad \lambda_2 = a_{11}a_{22} + a_{12}a_{21}.
\]
Moreover since
\[
\det A = a_{11}a_{22} - a_{12}a_{21} \quad \text{and} \quad \operatorname{per}(A) = a_{11}a_{22} + a_{12}a_{21}
\]
we find
\[
\lambda_1 = \det A \quad \text{and} \quad \lambda_2 = \operatorname{per}(A).
\]
Similarly it is easily seen that the eigenvalues of \(\varphi_T(A)\) are \(\lambda_1 = \det A\) and \(\lambda_2 = \operatorname{per}(A)\).
So the theorem is proved.

**Remark 2.** Unfortunately Theorem 2 is wrong for \(n \geq 3\). But \(\det A\) is always an eigenvalue of \(\varphi_T(A)\) for \(n \geq 3\). We state this fact as theorem.

**Theorem 3.** For \(n \geq 3\), at least an eigenvalue of \(\varphi_T(A)\) is equal to \(\det A\).

**Proof.** We remark that
\[
\det A = a_{ij}\alpha_{ij} + a_{i2}\alpha_{i2} + \cdots + a_{in}\alpha_{in}, \quad (i = 1, 2, \ldots, n)
\]
or
\[
\det A = a_{ij}\alpha_{ij} + a_{j2}\alpha_{j2} + \cdots + a_{nj}\alpha_{nj}, \quad (j = 1, 2, \ldots, n)
\]
where \(\alpha_{ij}\) denotes cofactor of \(a_{ij}\), i.e., \(\alpha_{ij} = (-1)^{i+j}\det(A_{ij})\).

Using the properties of determinants and considering (1.2) we have
\[
\det(\lambda I - \varphi_T(A)) = \det \begin{bmatrix}
\lambda - a_{11} & -a_{12} & \cdots & -a_{1n}\alpha_{1n} \\
-a_{21} & \lambda - a_{22} & \cdots & -a_{2n}\alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn}\alpha_{nn}
\end{bmatrix}
\]
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\[
\begin{bmatrix}
\lambda - \sum_{j=1}^{n} a_{1j} \alpha_{1j} & -a_{12} \alpha_{12} & \cdots & -a_{1n} \alpha_{1n} \\
\lambda - \sum_{j=1}^{n} a_{2j} \alpha_{2j} & \lambda - a_{22} \alpha_{22} & \cdots & -a_{2n} \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda - \sum_{j=1}^{n} a_{nj} \alpha_{nj} & -a_{n2} \alpha_{n2} & \cdots & \lambda - a_{nn} \alpha_{nn}
\end{bmatrix}
\]

\[
= \det
\begin{bmatrix}
\lambda - \det A & -a_{12} \alpha_{12} & \cdots & -a_{1n} \alpha_{1n} \\
\lambda - \det A & \lambda - a_{22} \alpha_{22} & \cdots & -a_{2n} \alpha_{2n} \\
& \vdots & \ddots & \vdots \\
\lambda - \det A & -a_{n2} \alpha_{n2} & \cdots & \lambda - a_{nn} \alpha_{nn}
\end{bmatrix}
\]

\[
= (\lambda - \det A) \det
\begin{bmatrix}
1 & -a_{12} \alpha_{12} & \cdots & -a_{1n} \alpha_{1n} \\
1 & \lambda - a_{22} \alpha_{22} & \cdots & -a_{2n} \alpha_{2n} \\
& \vdots & \ddots & \vdots \\
1 & -a_{n2} \alpha_{n2} & \cdots & \lambda - a_{nn} \alpha_{nn}
\end{bmatrix}
\]

it follows that \( \det A \) is eigenvalue of \( \varphi_T (A) \).

**Theorem 4.** Let \( A \) be \( n \times n \) real matrix, then all the row sums and column sums of \( \varphi_T (A) \) are equal to \( \det A \).

**Proof.** If \( r_1, r_2, \ldots, r_n \) denote the row sums of \( \varphi_T (A) \), then we write

\[
r_i = \sum_{j=1}^{n} a_{ij} \alpha_{ij}.
\]

On the other hand considering (1.2) we have

\[
\det A = r_i = \sum_{j=1}^{n} a_{ij} \alpha_{ij} \quad (i = 1, 2, \ldots, n).
\]

Similarly if \( c_1, c_2, \ldots, c_n \) denote the column sums of \( \varphi_T (A) \), then we write

\[
c_j = \sum_{i=1}^{n} a_{ij} \alpha_{ij}.
\]
Again considering (1.3) we have

\[ \det A = c_j = \sum_{i=1}^{n} a_{ij} \alpha_{ij} \quad (j = 1, 2, \ldots, n). \]

Thus the proof is complete.

**Corollary 1.** For \( A = (a_{ij}) \in M_n(R) \),

\[ e^T \mathcal{V}(A)e = n \det A, \]

where \( e = (1, 1, \ldots, 1)^T \).

**Proof.** We write

\[ e^T \mathcal{V}(A)e = \sum_{i,j=1}^{n} a_{ij} \alpha_{ij} = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij} \alpha_{ij} \right) \]

\[ = \sum_{i=1}^{n} \det A = n \det A \]

and the proof is complete.

**Theorem 5.** For any matrix \( A = (a_{ij}) \in M_n(R) \), the following statements are true.

(i) \[ \| \mathcal{V}(A) \|_1 \geq \| \det A \| \]

(ii) \[ \| \mathcal{V}(A) \|_{\infty} \geq \| \det A \| \]

(iii) \[ \| \mathcal{V}(A) \|_{\infty} \geq \| \det A \| \]

(iv) \[ \| \mathcal{V}(A) \|_1 \geq n \| \det A \| \]

**Proof.** (i) Considering the definition 3 and triangle inequality we have

\[ \| \mathcal{V}(A) \|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} | a_{ij} \alpha_{ij} | \geq \max_{1 \leq j \leq n} \left| \sum_{i=1}^{n} a_{ij} \alpha_{ij} \right| = \max_{1 \leq j \leq n} | \det A | = | \det A |. \]

(ii) By the definition 4 and triangle inequality, we write

\[ \mathcal{V}(A) \geq \det A \]

(iii) Considering the definition 5 and triangle inequality

\[ \mathcal{V}(A) \geq n \det A \]

it follows that

\[ \mathcal{V}(A) \geq \det A \]

\[ \mathcal{V}(A) \geq n \det A \]

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(iv) Similarly by the definition 6 and triangle inequality we have??

EMBED Equation.3??

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