Some Inequalities on The Permanents

Dursun TAŞCI [1]

Abstract: In this paper we obtained some inequalities about permanents of Hadamard product of matrices and permanents of sum of matrices.

Key Words: Permanent, Hadamard product, Positive Semidefinite Hermitian matrices

Permanentler Üzerine Bazı Eşitsizlikler

Özet: Bu çalışmada matrislerin Hadamard çarpımının Permanentleri ve matrislerin toplamının Permanentleri ile ilgili bazı eşitsizlikler elde edildi.

Anahtar Kelimeler: Permanent, Hadamard Çarpımı, Pozitif Yarı tanımlı Hermityen Matrisler.

Introduction and the Main Results

Definition 1.[1] The Permanent of real n×n matrix $A=(a_{ij})\in M_n$ is defined by

$$\mathrm{per}(A)=\sum_{\sigma\in S_n}\prod_{i=1}^{n}a_{i\sigma(i)} ,$$

where S_n is the symmetric group of order n.

The permanent can thus be thought of as a function whose domain is the set of n×n real matrices and whose range is the set of real numbers.

Definition 2.[2] If $A=(a_{ij})$ and $B=(b_{ij})$ are n×n matrices then their Hadamard product is the n×n matrix $C=A\circ B$ whose (i,j) entry is $a_{ij}b_{ij}$.

Lemma 1.[2] If A and B positive semidefinite Hermitian matrices then so is $A\circ B$.

Theorem 1.[1] If $A=(a_{ij})$ is an n×n matrix then for any i, $1\le i\le n$,

$$\mathrm{per}(A)=\sum_{j=1}^{n}a_{ij}\,\mathrm{per}(A_{ij}),$$

where A_{ij} denotes the submatrix obtained from A by deleting rows i and colums j.

[1] Selcuk University, Department of Mathematics, [42031]Campus/Konya/TURKEY

Theorem 2. Let $A \in M_n$ be positive semidefinite Hermitian matrix and define

$$\mu(A) = \begin{cases} \dfrac{\mathrm{per}(A)}{\mathrm{per}(A_{11})} & , \text{ if } A_{11} \text{ is positive semidefinite Hermitian} \\ 0 & , \text{ otherwise} \end{cases}$$

where A_{11} is the (n-1)×(n-1) principal submatrix of A that results from deleting the first row and column of A and M_n denotes n×n matrices. Then

$$\mu(A \circ B) \geq \mu(A) b_{11} + \mu(B) a_{11} - \mu(A)\mu(B) \tag{1}$$

Proof. It suffices to prove that

$$\frac{\mathrm{per}(A \circ B)}{\mathrm{per}(A_{11} \circ B_{11})} - \frac{\mathrm{per}(A)}{\mathrm{per}(A_{11})} b_{11} - \frac{\mathrm{per}(B)}{\mathrm{per}(B_{11})} a_{11} + \frac{\mathrm{per}(A)}{\mathrm{per}(A_{11})} \frac{\mathrm{per}(B)}{\mathrm{per}(B_{11})} \geq 0. \tag{2}$$

We have

$$\frac{\mathrm{per}(A \circ B)}{\mathrm{per}(A_{11} \circ B_{11})} - \frac{\mathrm{per}(A)}{\mathrm{per}(A_{11})} b_{11} - \frac{\mathrm{per}(B)}{\mathrm{per}(B_{11})} a_{11} + \frac{\mathrm{per}(A)}{\mathrm{per}(A_{11})} \frac{\mathrm{per}(B)}{\mathrm{per}(B_{11})} = \left(\frac{\mathrm{per}(A \circ B)}{\mathrm{per}(A_{11} \circ B_{11})} - a_{11} b_{11} \right)$$

$$+ \left(\frac{\mathrm{per}(A)}{\mathrm{per}(A_{11})} - a_{11} \right) \left(\frac{\mathrm{per}(B)}{\mathrm{per}(B_{11})} - b_{11} \right). \tag{3}$$

Now we must show that

$$\frac{\mathrm{per}(A \circ B)}{\mathrm{per}(A_{11} \circ B_{11})} - a_{11} b_{11} \geq 0, \tag{4}$$

$$\frac{\mathrm{per}(A)}{\mathrm{per}(A_{11})} - a_{11} \geq 0, \tag{5}$$

and

$$\frac{\mathrm{per}(B)}{\mathrm{per}(B_{11})} - b_{11} \geq 0, \tag{6}$$

respectively. Considering Theorem 1 we have

$$\mathrm{per}(A \circ B) = a_{11} b_{11} \mathrm{per}(A_{11} \circ B_{11}) + a_{12} b_{12} \mathrm{per}(A_{12} \circ B_{12}) + \cdots + a_{1n} b_{1n} \mathrm{per}(A_{1n} \circ B_{1n}), \tag{7}$$

where A_{1j} and B_{1j}, $1 \leq j \leq n$, denote the submatrices obtained from A and B by deleting row 1 and columns j respectively. Now from (7) we write

$$\mathrm{per}(A \circ B) \geq a_{11} b_{11} \mathrm{per}(A_{11} \circ B_{11})$$

or

$$\frac{\mathrm{per}(A \circ B)}{\mathrm{per}(A_{11} \circ B_{11})} - a_{11} b_{11} \geq 0.$$

Similarly the inequalities (5) and (6) are satisfied . From (4), (5), and (6), the inequality (1) holds and thus the proof is complete.

Theorem 3. If $A_1, A_2, \ldots, A_n$ are n×n matrices with nonnegative entries then

$$\operatorname{per}\left(\sum_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} \operatorname{per}(A_i). \tag{8}$$

Proof. We use induction on n for the proof of Theorem. It is true for n = 2 . Indeed the (i,j) entry of $A_1 + A_2$ is just $a_{ij}^{(1)} + a_{ij}^{(2)}$, where $A_1 = (a_{ij}^{(1)})$ and $A_2 = (a_{ij}^{(2)})$ are $n \times n$ matrices with nonnegative entries. Thus a typical term in the sum defining $\operatorname{per}(A_1 + A_2)$ is

$$\prod_{i=1}^{n} \left(a_{i\sigma(i)}^{(1)} + a_{i\sigma(i)}^{(2)}\right). \tag{9}$$

Now if we multiply out the product (9) and throw a way all terms expect

$$\prod_{i=1}^{n} a_{i\sigma(i)}^{(1)} \quad \text{and} \quad \prod_{i=1}^{n} a_{i\sigma(i)}^{(2)}$$

we obtain (remember A_1 and A_2 have nonnegative entries)

$$\prod_{i=1}^{n} \left(a_{i\sigma(i)}^{(1)} + a_{i\sigma(i)}^{(2)}\right) \geq \prod_{i=1}^{n} a_{i\sigma(i)}^{(1)} + \prod_{i=1}^{n} a_{i\sigma(i)}^{(2)} . \tag{10}$$

If we sum all the inequalities (10) for $\sigma \in S_n$ we get

$$\sum_{\sigma \in S_n} \prod_{i=1}^{n} \left(a_{i\sigma(i)}^{(1)} + a_{i\sigma(i)}^{(2)}\right) \geq \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}^{(1)} + \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}^{(2)} ,$$

that is,

$$\operatorname{per}(A_1 + A_2) \geq \operatorname{per}(A_1) + \operatorname{per}(A_2).$$

We assume now that the inequality (8) is true for n-1 and show that assumption implies that (8) holds for n. Now, if

$$\operatorname{per}\left(\sum_{i=1}^{n-1} A_i\right) \geq \sum_{i=1}^{n-1} \operatorname{per}(A_i)$$

then

$$\operatorname{per}\left(\sum_{i=1}^{n} A_i\right) = \operatorname{per}\left(\sum_{i=1}^{n-1} A_i + A_n\right) \geq \operatorname{per}\left(\sum_{i=1}^{n-1} A_i\right) + \operatorname{per}(A_n)$$

$$\geq \sum_{i=1}^{n-1} \operatorname{per}(A_i) + \operatorname{per}(A_n) = \sum_{i=1}^{n} \operatorname{per}(A_i)$$

thus we have proved by induction that the inequality (8) holds for all n.

Corollary 1. If A is $n \times n$ matrix with nonnegative entries and

$$H(A) = \frac{A + A^T}{2}$$

then

$$\operatorname{per}(H(A)) \geq 2^{1-n} \operatorname{per}(A),$$

where A^T denotes the transpose of A.

Proof. By the Theorem 3 we have

$$\mathrm{per}(H(A)) = \mathrm{per}\left(\frac{A + A^T}{2}\right) = \frac{1}{2^n}\,\mathrm{per}\left(A + A^T\right)$$

$$\geq \frac{1}{2^n}\left(\mathrm{per}(A) + \mathrm{per}\left(A^T\right)\right)$$

$$= \frac{1}{2^n}\left(2\,\mathrm{per}(A)\right)$$

$$= \frac{1}{2^{n-1}}\,\mathrm{per}(A)$$

thus the proof is complete.

Corollary 2. If A and B are n×n matrices with nonnegative entries then

$$[\mathrm{per}(A + B)]^2 \geq 4\,\mathrm{per}(A)\mathrm{per}(B).$$

Proof. Using arithmetic-geometric mean inequality and considering Theorem 3 we have

$$\left[\frac{\mathrm{per}(A + B)}{2}\right]^2 \geq \left[\frac{\mathrm{per}(A) + \mathrm{per}(B)}{2}\right]^2 \geq \mathrm{per}(A)\mathrm{per}(B)$$

and therefore we write

$$[\mathrm{per}(A + B)]^2 \geq 4\,\mathrm{per}(A)\mathrm{per}(B).$$

We conclude the paper with a theorem.

Theorem 4. If A and B are n×n matrices with nonnegative entries and $A \geq B$ then

$$\mathrm{per}(A) - \mathrm{per}(B) \geq \mathrm{per}(A - B) \tag{11}$$

and

$$\left|\mathrm{per}(A - B)\right| = \left|\mathrm{per}(B - A)\right| . \tag{12}$$

Proof. By Theorem 3 we write

$$\mathrm{per}(A) = \mathrm{per}(A - B + B) \geq \mathrm{per}(A - B) + \mathrm{per}(B)$$

and it follows that the inequality (11) holds. On the other hand we have

$$\left|\mathrm{per}(A - B)\right| = \left|\mathrm{per}(-(B - A))\right| = \left|(-1)^n\,\mathrm{per}(B - A)\right| = \left|\mathrm{per}(B - A)\right|$$

Thus the proof is complete.

References

1- Minc, H., Permanents, **In Encyclopaedia of Mathematics and Its Applications**, Vol. 6, Addison-Wesley, Reading MA (1978)

2- Horn, R.A., Johnson, C:R., **Topics in Matrix Analysis**, Cambridge University Press. (1991)